Indian Statistical Institute, Bangalore
B.Math (Hons.) II Year, First Semester
Semestral Examination
Analysis III

Time: 3 hours 2 Dec 2011 Instructor: Pl. Muthuramalingam
Maximum marks: 50

Note: The paper has two part, Part A and Part B. You can get a maximum of 45 (forty five only) in part A.

Part A

1. State and prove Stokes Theorem. [7]

2. Give an example of a continuous function $f : (0,1) \times (0,1) \to R$ such that $\int_0^1 dyf(x,y), \int_0^1 dx \int_0^1 dyf(x,y), \int_0^1 dxf(x,y), \int_0^1 dy \int_0^1 dxf(x,y)$ all exist, $\int_0^1 dy \int_0^1 dxf(x,y) \neq \int_0^1 dx \int_0^1 dyf(x,y)$ and prove your claim. [3]

3. Let $g : [-1,1] \to R$ be any continuous even function i e $g(t) = g(-t)$ for all t in $[-1,1]$. Let $\mathbb{E} = lin.$ span $\{1, t^2, t^4, \cdots, t^{2k}, \cdots\}$. Show that for each $\delta > 0$ there exists p_δ in $\mathbb{E}$ such that $\sup\limits_{-1 \leq t \leq 1} | g(t) - p_\delta(t) | \leq \delta$. [3]

4. Let $h \varepsilon C[a,b]$ be differentiable and $h' \varepsilon C[a,b]$. Show that there exists a sequence $P_1, P_2, \cdots, P_n, \cdots$ of polynomials such that

$$0 = \lim_{n \to \infty} \sup_{a \leq t \leq b} [| P_n(t) - h(t) | + | P_n'(t) - h'(t) |]$$

. [4]

5. Let $f_1, f_2, \cdots \varepsilon C[a,b], f_n$ is differentiable and $f_n' \varepsilon C[a,b]$. Let $g, h \varepsilon C[a,b]$ be such that $f_n \to g$ and $f_n' \to h$ uniformly on $[a,b]$. Then show that g is differentiable. [4]

6. Let $B_n(r) = \{(x_1, x_2, \cdots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 \leq r^2\}$.
Let $v_n(r) = \int\limits_{B_n(r)} dx_1 dx_2 \cdots dx_n$.

(a) Find a relation between $v_n(r)$ and $v_n(1)$.

(b) For $k \geq 3$ find a relation between $v_k(1)$ and $v_{k-2}(1)$. [4]

7. Let $f_k : (0,\infty) \times R \times R \to R$ be given by

$$f_k(t,x,y) = e^{-tk^{\cdot 001}} \sin(k^{\cdot 002}x) \cos(k^{\cdot 003}y)$$

.

Define $f : (0,\infty) \times R \times R \to R$ by $f(t,x,y) = \sum\limits_{k=1}^{\infty} f_k(t,x,y)$.

(a) Show that RHS is summable.

(b) Show that f is a continuous function. [4]

8. Let $\mathbf{F}(x,y,z) = (x^2, -2xy, 3xz)$,
$V = \{(x,y,z) : x \geq 0, y \geq 0, z \geq 0, x^2 + y^2 + z^2 \leq 4\}$.
Evaluate $\int\int\limits_V\int div\mathbf{F}$. [3]

9. Let $\mathbf{F}$ be as in question 8. Let $S = \{(x,y,z) : z \geq 0, y \geq 0, z \geq 0, x^2 + y^2 + z^2 = 4\}$.
Find $\int\limits_S \mathbf{F}$. Note that S is a portion of the boundary ∂V of V of question 8. [7]

10. Verify Stokes theorem for $\int\limits_{\lceil} zdx - xdz$

where

$\lceil=$ boundary of S,

$$\begin{aligned} S = \{(x,y,z) : x^2 + y^2 + z^2 &= 4a^2, \\ x^2 + y^2 &\leq 2ax, \\ x > 0, y > 0, z &> 0\} \end{aligned}$$

where $a > 0$. [10]

11. Let $a > 0, b > 0, k > 0,$ h real and $ab - h^2 > 0$. It is known that $\{(x,y) : ax^2 + 2hxy + by^2 \leq k\}$ is an elliptic disk. Find its area in terms of $k, ab - h^2$. [4]

Part B

12. Let $g_o : [-1,1] \to R$ be the continuous function given by $g_o(t) =\mid t \mid$. Let $Q_1, Q_2, \cdots$ be any sequence of polynomials converging uniformly to g_0 on $\lceil -1, 1]$. Fix $\delta > 0$. For each $k = 1,2,3,\cdots$ define $b_k = \sup\limits_{n,j \geq k} \sup\limits_{|t| \leq \delta} \mid Q_n'(t) - Q_j'(t) \mid$ show that $\liminf\limits_k b_k > 0$ [Here δ is fixed]. [5]